Modeling and Simulation

From observation via implementation to computation and back…

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Part Seven

Error Propagation

Sources of Errors

- Unlike symbolic calculations: Errors are *inherent* in numerical calculations
  - Modeling
  - Data
  - Computational
    - Round-off
    - Truncation
    - Bugs
    - Hardware errors

Computational error is a necessary evil!

- Round-off 9.999 10^9 and 9.999 10^{-9}
  - Sum: 22 digits: 9999000000,000000009999
  - Prod: 8 digits: 99.980001
- Truncation

$$e^x \approx \sum_{n=0}^{100} \frac{x^n}{n!}$$

or for example:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Definitions

- Error := Approximation - RealValue
- Relative Error := | Error/RealValue|
- => Approximation := RealValue (1+RE)

$$\tilde{x} = x(1+\varepsilon)$$

More definitions

f(x) : Maps Input to Output

$$\text{Propagated data error}: f(\tilde{x}) - f(x)$$

Even worse since we have $\tilde{f}(\tilde{x})$

So we can calculate the *total* error:

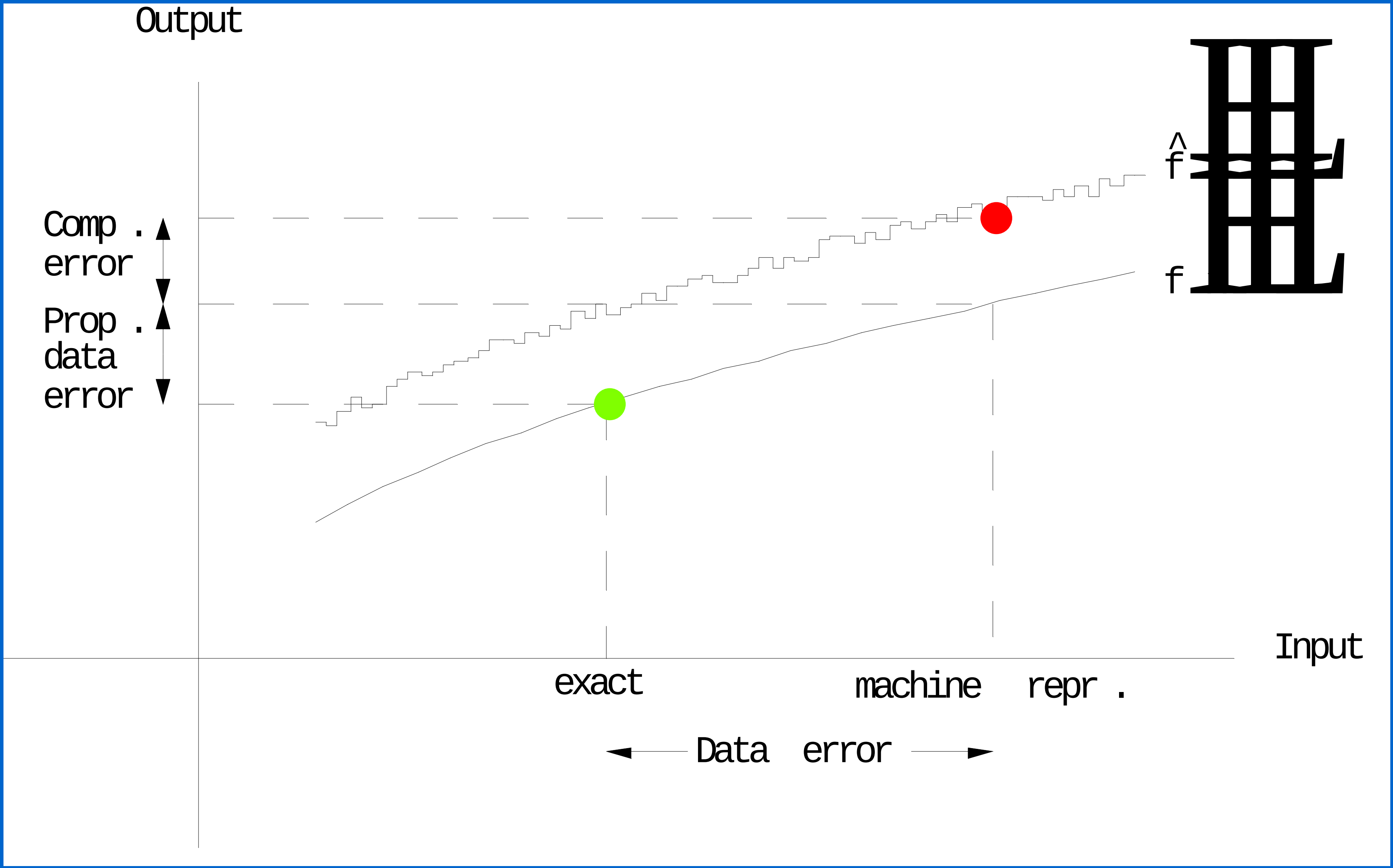
Total Error

$$\text{Total error} \approx \left| \tilde{f}(\tilde{x}) - f(x) \right| = \left| \tilde{f}(\tilde{x}) - f(\tilde{x}) + f(\tilde{x}) - f(x) \right|$$

$$\leq \left| \tilde{f}(\tilde{x}) - f(\tilde{x}) \right| + \left| f(\tilde{x}) - f(x) \right|$$

$\Uparrow$ *Computational Error (Implementation)*

$\Uparrow$ *Propagated data error (Condition)*


Output
Comp.
error
Prop.
data
error
f̂
f
Input
exact
machine repr.
Data error

Conditioning

- Example: Chaotic systems: Extreme sensitive to initial conditions
  - Gilplin's model of Predator/Prey
  - Weather predictions (< 15 days…)

$$\text{Sensitivity} \equiv \text{Condition Number} \equiv \frac{|\text{Relative error in result}|}{|\text{Relative error in data}|} >> 1$$

- Ill-conditioned : Sensitive
- Well-Conditioned : in-Sensitive

Calculate Condition Number of Unary Operation:

- Assume f(x) is linear (else use Taylor Expansion)
- Assume *Exact* calculation
- Assume approximation result of f(x) is due to use of approximated x
- Use: Appr = True (1+RelError)

Consequence:

$$\tilde{f}(x) \equiv f(x)\,(1+\varepsilon_{f(x)}) \equiv f(x(1+\varepsilon_x))$$

Therefore we have:

$$f(x)\varepsilon_{f(x)} = f\left(x + x\varepsilon_x\right) - f\left(x\right)$$

$$\Rightarrow \frac{\varepsilon_{f(x)}}{\varepsilon_x} \equiv \text{Sensitivity} = \frac{f\left(x + x\varepsilon_x\right) - f\left(x\right)}{f(x)\varepsilon_x}$$

So if derivative of $f\left(x\right)$ in x exists than:

$$\frac{f\left(x+x\varepsilon_x\right)-f\left(x\right)}{x\varepsilon_x} \xrightarrow{\varepsilon_x\to 0} f'\left(x\right)$$

$$\text{Substitute}: \frac{\varepsilon_{f(x)}}{\varepsilon_x} \to \frac{xf'\left(x\right)}{f\left(x\right)} \text{ for } \varepsilon_x \to 0$$

Then the Condition Number is a measure of the relative sensitivity of $f(x)$ for small variations in x:

$$\left|\frac{\varepsilon_{f(x)}}{\varepsilon_x}\right| \equiv \text{Condition number of } f(x) \text{ in } x$$

Example I: Condition Number of e^x

Relative derivative is $\dfrac{x.e^x}{e^x} = x \Rightarrow \text{CN} = |x|$

Consequence I: Exponential function is sensitive for $|x| >> 1$

insensitive for $|x| << 1$

Example II : Condition Number of $\ln(x)$

$$\text{Relative derivative of } \ln(x) \equiv \frac{x(\ln(x))'}{\ln(x)} = \frac{1}{\ln(x)}$$

$$\Rightarrow \text{CN} = \left|\frac{1}{\ln(x)}\right|$$

Consequence : $\ln(x)$ is sensitive for $x \approx 1$